Quadimodularity of the kth Residual Cranks

Special Session on Quadratic Forms, Theta Functions, and Modularity

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2022 April 7

Agenda

- Preliminaries
- Why Partitions?
- A Combinatorial Touch
- Now About That Quasimodularity

Definition

Let $q = \exp(2\pi i z)$. For $k \geq 1$, the *Eisenstein series* $E_{2k}(z)$ is

$$E_{2k}(z) = 1 + \frac{(2\pi)^{2k}}{(-1)^k (2k-1)! \zeta(2k)} \Phi_{2k-1}(q),$$

where

$$\Phi_\ell(q) = \sum_{n=1}^{\infty} \left\{ \sum_{d|n} d^\ell \right\} q^n$$

is a divisor sum generating function.

Recall, quasimodular forms are generated as products of the E_{2k}.

Definition

Let $\overline{\mathcal{W}}_k(\Gamma)$ be the space of all quasimodular forms of weight at most $2k$, which transform under Γ, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and which have no constant term.

We are interested in spaces of the form

$$f(z)\overline{\mathcal{W}}_k(\Gamma) := \{f(z)g(z) \mid g(z) \in \overline{\mathcal{W}}_k(\Gamma)\}.$$

Definition

A *partition* π is a non-increasing sequence of positive integers. If the sum of these integers is n, then we write $\pi \vdash n$, or $|\pi| = n$. Let $p(n)$ denote the number of partitions of n.

The partitions $\pi \vdash 4$ are

$$\begin{array}{ll} (4) & (3,1) \\ (2,2) & (2,1,1) \\ (1,1,1,1). & \end{array}$$

Thus, $p(4) = 5$.

Take the generating function for partitions,

$$P(q) = \sum_{n=0}^{\infty} p(n) q^n.$$

Euler proved that

$$P(q) = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}.$$

We will abbreviate these kinds of products using the q-Pochhamer symbol

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - a q^i).$$

Thus $P(q) = 1/(q; q)_\infty$.

Why then are partitions related to quasimodularity? Let $\delta_q = q\frac{\partial}{\partial q}$.

Then logarithmic differentiation shows that $\delta_q P$ may be written in terms of the divisor sum fuctions Φ_ℓ, since

$$\delta_q \frac{1}{1-q^i} = \frac{iq^i}{1-q^i} = \sum_{n=1}^{\infty} iq^{in}.$$

Crank Function

Definition

If a partition π does not contain any 1s, then the *crank* of π is defined to be the largest part of π.
Otherwise, let $w(\pi)$ denote the number of 1's occurring in π, and let $\mu(\pi)$ denote the number of parts of π which are larger than $w(\pi)$. In this case, the crank of π is defined to be

$$c(\pi) = \mu(\pi) - w(\pi).$$

Payoff: The crank is an integer-valued function which illuminates the structure of partitions.

Take $M_j(n)$ to be the jth moment of the crank,

$$M_j(n) = \sum_{\pi \vdash n} c(\pi)^j.$$

Theorem (Dyson, 1989)

For $n \geq 0$,

$$M_2(n) = 2np(n).$$

That is, the $M_2(n)$ are the coefficients of $2\delta_q P$. The combinatorial interpretation is that $M_2(n)$ counts the total size of parts across all partitions of n.

Theorem (Jennings-Shaffer, 2015)

The following crank functions are in $(-q;q^2)_\infty P(q^2)\mathcal{W}_k(\Gamma_0(4))$*:*

$$\delta_q^m C1_{2j},\ \textit{for } m \geq 0, 1 \leq j \leq k, j+m \leq k$$
$$\delta_q^m C2_{2j},\ \textit{for } m \geq 0, 1 \leq j \leq k, j+m \leq k$$
$$\delta_q^m C4_{2j},\ \textit{for } m \geq 0, 1 \leq j \leq k, j+m \leq k$$

We aim to bring these results to a more general setting.

Overpartitions

Definition

A *overpartition* is a non-increasing sequence of positive integers, where the first occurrence of each part may be overlined.

The overpartitions $\pi \vdash 3$ are

$$\begin{array}{llll} (3) & (\overline{3}) & (1,1,1) & (\overline{1},1,1) \\ (2,1) & (2,\overline{1}) & (\overline{2},1) & (\overline{2},\overline{1}). \end{array}$$

The generating function for overpartitions is

$$\overline{P}(q) = \sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = (-q;q)_\infty P(q).$$

Residual Cranks

Definition

For $k \geq 1$, the kth residual partition of π is a partition π' consisting of $1/k$th of each of the non-overlined parts of π that are divisible by k. The kth *residual crank* of π is then defined to be $c_k(\pi) = c(\pi')$.

For example,

$$
\begin{aligned}
c_1((4,\overline{3},2)) &= c((4,2)) = 4\\
c_2((4,\overline{3},2)) &= c((2,1)) = 0\\
c_3((4,\overline{3},2)) &= c(\emptyset) = 0\\
c_4((4,\overline{3},2)) &= c((1)) = -1\\
c_5((4,\overline{3},2)) &= c(\emptyset) = 0.
\end{aligned}
$$

Definition

Let $nov_k(n)$ denote the sum of all non-overlined parts which vanish modulo k, taken across all overpartitions $\pi \vdash n$.

We see that $nov_2(3) = 4$

$$\begin{array}{llll} (3) & (\overline{3}) & (1,1,1) & (\overline{1},1,1) \\ (2,1) & (2,\overline{1}) & (\overline{2},1) & (\overline{2},\overline{1}). \end{array}$$

Theorem (M., Simonič, 2021)

Take $\overline{M[k]}_\ell(n)$ to be the ℓth moment of the kth residual cranks. For $n \geq 0$ and $k \geq 1$,

$$k \cdot \overline{M[k]}_2(n) = 2 \cdot nov_k(n).$$

Sketch.

The two variable generating function for the kth residual crank is given by

$$\overline{C[k]} := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(q^k, q^k; q^k)_\infty}{(zq^k, q^k/z; q^k)_\infty}.$$

We can calculate the generating function for $\overline{M[k]_2}$ by using $\delta_z = z\frac{\partial}{\partial z}$, since

$$\delta_z^2 \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} m^2 M(m,n) z^m q^n.$$

Substituting $z = 1$ completes the calculation.

For our series,

$$\delta_z^2 \frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(q^k,q^k;q^k)_\infty}{(zq^k,q^k/z;q^k)_\infty}$$
$$= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \delta_z^2 \frac{(q^k;q^k)_\infty}{(zq^k,q^k/z;q^k)_\infty}$$
$$= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \delta_z^2 C(z;q^k),$$

where $C(z;q)$ is the two variable generating function for the partition crank.

The problem reduces to Andrews' result for ordinary partitions.

$$\begin{aligned}\sum_{n=0}^{\infty} k\cdot\overline{M[k]}_2(n)q^n &= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty} k\cdot M_2(n)q^{kn}\\ &= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty} 2kn\cdot p(n)q^{kn}\\ &= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{\pi'} 2k|\pi'|q^{k|\pi'|}\\ &= \sum_{n=0}^{\infty} 2\cdot nov_k(n)q^n. \quad \square\end{aligned}$$

Theorem (Bringman, Lovejoy, Osburn; 2009)

The following crank functions are in $\overline{P}\mathcal{W}_k(\Gamma_0(2))$*:*

$$\delta_q^m \overline{C[1]}_{2j}, \text{ for } m \geq 0, 1 \leq j \leq k, j+m \leq k$$
$$\delta_q^m \overline{C[2]}_{2j}, \text{ for } m \geq 0, 1 \leq j \leq k, j+m \leq k$$

Let $\overline{C[k]}_j(q)$ denote the jth moment generating function for the kth residual crank.

Theorem (M., Simonič, 2021)

For $j, k \geq 1$ and $m \geq 0$, the function

$$\delta_q^m \overline{C[k]}_{2j}(q)$$

is in the space $\frac{(-q;q)_\infty}{(q;q)_\infty} \mathcal{W}_{j+m}(\Gamma_0(lcm(2,k)))$.

That is, $\delta_q^m \overline{C[k]}_{2j}(q)$ is $\overline{P}(q)$ times a quasimodular form.

Thank you!